Homework 3

2.1 **#3** $N(T) = \{(a_1,a_2) \mid a_1+a_2=0,\ 2a_1-a_2=0\} = \{(0,0)\}$

$\therefore \dim R(T) = 2$

$w \in R(T) \quad w = T(a_1,a_2) = (a_1+a_2, 2a_1-a_2)$

$= (a_1, 2a_1) + (a_2, -a_2)$

$= a_1(1,2) + a_2(1,-1)$

Bases for $R(T)$: $(1,2), (1,-1)$

#5 $f(x) = a_2x^2 + a_1x + a_0$

$f'(x) = 2a_2x + a_1$

$f'(x) + xf(x) = a_2x^3 + a_1x^2 + (2a_2+a_0)x + a_1$

$\therefore$ in $N(T)$ if $a_2=0,\ a_1=0,\ a_0=0$

$\therefore N(T) = \{0\}$.

Given $b_3x^3 + b_2x^2 + b_1x + b_0 = T(f(x))$

$a_2 = b_3,\quad a_1 = b_2,\quad (2a_2+a_0) = b_1,\quad a_1 = b_0,\quad a_0$ arbitrary

$T(f(x)) = a_2x^3 + a_1x^2 + (2a_2+a_0)x + a_1$

$= a_2(x^3+2x) + a_1(x^2+1) + a_0x$

Bases for $R(T)$: $x^3+2x,\ x^2+1,\ x$

#6 T is onto $\quad a \in F$

$T\begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} = a \quad$ Basis for $R(T)$ is $1 \in F$

$\dim N(T) = n^2-1 \quad \operatorname{tr}(A) = 0 \quad a_{11} + \cdots + a_{nn} = 0$

$A = a_{11}E_{11} + \cdots a_{n-1\,n-1}E_{n-1\,n-1} + (-a_{11} - \cdots - a_{n-1\,n-1})E_{nn} +$

$\sum_{i \neq j} a_{ij}E_{ij}$

Basis of $N(T)$: Consider

$$A = a_{11}(E_{11} - E_{nn}) + a_{22}(E_{22} - E_{nn}) + \cdots + a_{n-1\,n-1}(E_{n-1\,n-1} - E_{nn})$$
$$+ \sum_{i \neq j} a_{ij} E_{ij}$$

#10 $T: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad (2,3) = x(1,0) + y(1,1)$

$x + y = 2 \quad y = 3 \quad \therefore x = -1$

$T(2,3) = -1\, T(1,0) + 3\, T(1,1) = -1(1,4) + 3(2,5) = (5,11)$

Yes Since $(1,4)$, $(2,5)$ is a basis T is onto

$\therefore \dim N(T) = 0 \quad N(T) = \{0\}$.

#17 (a) $\underset{m}{\dim V} < \underset{m}{\dim W} \quad r = \text{rank } T, \quad p = \text{null } T$

$r + p = n < m$

If T is onto, $r = m$ so $m + p < m \quad \therefore p < 0$ impossible

#21 (b) T is clearly onto and $T(a_1, a_2, \ldots) = T(a_1', a_2, \ldots)$ with $a_1 \neq a_1'$.

#28 $x \in R(T) \quad \therefore x = Ty, \quad Tx = T(Ty) \in R(T)$

$x \in N(T) \quad \therefore Tx = 0 \in N(T)$

#38 $T(z+w) = \overline{z+w} = \bar{z} + \bar{w} = Tz + Tw, \quad z, w \in \mathbb{C}$

Suppose T is linear $T(az) = a\, Tz \quad a, z \in \mathbb{C}$

so $\overline{az} = a\bar{z}$ But $\overline{az} = \bar{a}\bar{z}$

2.2 **#2b** $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

#4 $TE_{11} = 1 \quad TE_{12} = 1 + x^2 \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ See back of book for others

$TE_{21} = 0 \quad TE_{22} = 2x$

#8 $\beta = \{v_1, \ldots, v_n\} \quad v = a_1 v_1 + \cdots + a_n v_n, \; w = b_1 v_1 + \cdots + b_n v_n$

$v + w = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n, \quad av = (aa_1)v_1 + \cdots + (aa_n)v_n$

$$T(v+w) = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = T(v) + T(w)$$

$$T(av) = \begin{pmatrix} aa_1 \\ \vdots \\ aa_n \end{pmatrix} = a \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a\,T(v)$$